1 July 1999

Optics Communications

Optics Communications 165 (1999) 11–18

www.elsevier.com/locate/optcom

Phase space conservation and selection rules for astigmatic mode converters

James B. Murphy *

National Synchrotron Light Source, Brookhaven National Laboratory, Upton, NY 11973-5000, USA

Received 11 February 1999; received in revised form 13 May 1999; accepted 13 May 1999

Abstract

An astigmatic mode converter consists of a pair of cylindrical lenses which, when properly located and oriented, can transform a Hermite–Gauss mode (HG_{nm}) with rectangular symmetry into a Laguerre–Gauss mode (LG_{nm}) with cylindrical symmetry (see E. Abramochkin, V. Volstnikov, Opt. Commun. 83 (1991) 123; M.W. Beijersbergen, L. Allen, H.E.L.O. van der Veen, J.P. Woerdman, Opt. Commun. 96 (1993) 123; M.J. Padgett, J. Arlt, N. Simpson, Am. J. Phys. 64 (1996) 77; and J. Courtial, M.J. Padgett, Opt. Commun. 159 (1999) 13). In this paper we discuss how the conservation of phase space area determines the selection rules on which mode conversions are possible. A discussion of the beam quality factor M^2 is also given for both Hermite–Gauss and Laguerre–Gauss modes. © 1999 Published by Elsevier Science B.V. All rights reserved.

Keywords: Phase space; Mode conversion; Beam quality

1. Introduction

In Refs. [1–4], the authors describe a so-called astigmatic mode converter consisting of a pair of cylindrical lenses that will convert a monochromatic Hermite–Gauss mode (HG_{nm}) with rectangular symmetry into a Laguerre–Gauss mode (LG_{nm}) with cylindrical symmetry. We will use the notation of Ref. [2] to describe the normalized electric field of Hermite–Gauss and Laguerre–Gauss modes. The definitions are as follows:

$$u_{nm}^{HG}(x,y,z)=\frac{C_{nm}^{HG}}{w(z)}\exp\left[-\mathrm{i}k\frac{x^2+y^2}{2R(z)}\right]\exp\left[-\frac{x^2+y^2}{w^2(z)}\right]\cdot\exp[-\mathrm{i}(n+m+1)\psi(z)]\,H_n\left(\sqrt{2}\,\frac{x}{w(z)}\right)H_m\left(\sqrt{2}\,\frac{y}{w(z)}\right)\tag{1}$$

* Tel.: +1-516-344-5160; fax: +1-516-344-3029; e-mail: jbm@bnl.gov

PII: S0030-4018(99)00238-2

and

$$u_{nm}^{\mathrm{LG}}(x,y,z)=\frac{C_{nm}^{\mathrm{LG}}}{w(z)}\exp\left[-\mathrm{i}k\frac{r^2}{2R(z)}\right]\exp[-\mathrm{i}(n-m)\phi]\left(\sqrt{2}\,\frac{r}{w(z)}\right)^{|n-m|}$$
$$\cdot(-1)^{\min(n,m)}\exp\left[-\frac{r^2}{w^2(z)}\right]\exp[-\mathrm{i}(n+m+1)\psi(z)]\,L_{\min(n,m)}^{|n-m|}\left(2\frac{r^2}{w^2(z)}\right), \tag{2}$$

where $H_n(x)$ is the Hermite polynomial of order n and L_p^{ℓ} is the generalized Laguerre polynomial and

$$C_{nm}^{\mathrm{HG}}=\left(\frac{2}{\pi n!m!}\right)^{1/2}2^{-(n+m)/2},\ C_{nm}^{\mathrm{LG}}=\left(\frac{2}{\pi n!m!}\right)^{1/2}\min(n,m)!, \tag{3}$$

$$z_R\equiv\frac{\pi w_0^2}{\lambda}, \tag{4}$$

$$R(z)\equiv\frac{(z^2+z_R^2)}{z}, \tag{5}$$

$$\frac{1}{2}kw^2(z)\equiv\frac{(z^2+z_R^2)}{z_R}, \tag{6}$$

$$\psi(z)\equiv\arctan\left[\frac{z}{z_R}\right]. \tag{7}$$

In Ref. [2] it is shown that a HG_{nm} mode whose principal axes are rotated by 45° with respect to the first cylindrical lens of the astigmatic mode converter will be converted into a LG_{nm} with the same mode numbers (n,m). This result can be thought of as a ‘selection rule’ that limits which HG_{nm} mode is converted to which LG_{nm} mode. This brings to mind the question of what determines these selection rules.

In [2] the discussion focuses on the similarity of a 45° rotated HG_{nm} modes and a LG_{nm} when both are expanded in a complete set of Hermite–Gauss modes oriented along the principal axes. The astigmatic mode converter modifies the Guoy phase differently in the two principal planes and thereby converting the mode from Hermite–Gauss to Laguerre–Gauss in character with the same set of mode numbers (n,m). In this paper we will discuss another criteria that the selection rules must satisfy.

Since symplectic ABCD matrices can characterize the cylindrical lenses, the phase space area of the light beam must remain unchanged in traversing the mode converter. In this paper we will show that it is this principle of phase space area conservation that governs the selection rules for mode conversion. We begin with a discussion of the phase space of light beams and show how this can be used to determine the selection rules for astigmatic mode converters. In the process we will also discuss the laser beam quality factor M^2 that frequently appears in the literature.

2. Uncertainty relations and phase space area

Quantum mechanics restricts the product of the uncertainty in position and momentum for a particle in one dimension [5]

$$\Delta x\cdot\Delta p_x\geq\frac{\hbar}{2}, \tag{8}$$

where $(\Delta x)^2\equiv\int(x-\bar{x})^2|\Psi(x)|^2\mathrm{d}x$ and $(\Delta p_x)^2\equiv\int(p_x-\bar{p}_x)^2|\widetilde{\Psi}(p_x)|^2\mathrm{d}p_x$. The minimum uncertainty product, corresponding to the equal sign in the above equation, occurs for a Gaussian wave packet of a free particle.

The uncertainty product can be viewed as setting a lower bound on the (x, p_x) phase space area. In quantum mechanics the evolution of the particle is governed by unitary transformations that insure that the phase area is conserved as the system evolves in time. The shape of the occupied phase space may change but the area (or volume in higher dimensional systems) is an invariant [6].

For a monochromatic, two-dimensional light beam propagating in the z-direction the momentum is $\vec{p} = \hbar\vec{k}$. Thus the literal translation of the above uncertainty relation for the transverse coordinates becomes [7]

$$\Delta x \cdot \Delta k_x \cdot \Delta y \cdot \Delta k_y \geq \left(\frac{1}{2}\right)^2 . \tag{9}$$

If we define the angles, $\theta_x = k_x/k$ and $\theta_y = k_y/k$, and $k \equiv 2\pi/\lambda$, we can write Eq. (9) as

$$\Delta x \cdot \Delta \theta_x \cdot \Delta y \cdot \Delta \theta_y \geq \left(\frac{\lambda}{2}\right)^2 . \tag{10}$$

It can be seen that λ is for 'light' particles what $\hbar$ is for massive particles. In the case of light, the minimum uncertainty product occurs for a fundamental Gaussian mode HG_{00} with the uncertainties defined in terms of the second moments of the respective quantities.

If the propagation of the light beam is governed by symplectic ABCD matrices, such as simple drifts and thin lenses, the phase space area occupied by the light beam is an invariant [8]. The two cylindrical lenses that comprise the astigmatic mode converter can be represented by symplectic ABCD matrices so the phase space area of the 45 degree-rotated HG_{nm} mode at the input should equal that of the LG_{nm} mode at the output.

In what follows we shall discuss under what conditions Eq. (10) is correct and consider generalizations of this phase space conservation relation. First we will make contact with the beam quality factor, M^2, that has been used to characterize the phase space area for light beams [9]. We will then compute the phase space area for Hermite–Gauss and Laguerre–Gauss modes and show that its invariance governs the selection rules for mode conversion.

3. Beam quality factor

The beam quality factor M^2 was introduced in [9] to characterize how a given light beam differs from a fundamental Gaussian mode (HG_{00} or LG_{00}). For one dimension, Ref. [9] gives the following definition:

$$M_x^2 \equiv \frac{\text{Real beam space} - \text{bandwidth product}}{\text{Fundamental Gaussian beam space} - \text{bandwidth product}} . \tag{11}$$

In the terminology of this paper, the 'space-bandwidth product' is called the beam 'phase space area'. Thus M^2 is simply a measure of how much larger the phase space area is for a given beam as compared to the minimum area of a fundamental Gaussian mode. Additional discussion of the beam quality factor M^2 can be found in Refs. [10–16] and the citations therein.

Ref. [9] states that Hermite–Gauss modes HG_{nm} can be characterized by a pair of beam quality factors for the two transverse dimensions

$$M_x^2 = 2n+1, \quad M_y^2 = 2m+1, \tag{12}$$

or a single 'cylindrical quality factor'

$$M_r^2 = \frac{M_x^2 + M_y^2}{2} = n+m+1. \tag{13}$$

For Laguerre–Gauss modes LG_{nm} Ref. [9] defines the cylindrical quality factor as

$$M_r^2 = 2p + |\ell| + 1, \tag{14}$$

where we let $p \equiv \min(n,m)$ and $\ell \equiv n - m$ to rectify the notational differences between our paper and Ref. [9].

Which of these beam quality factors do we use to determine if the phase space area of the input HG_{nm} mode to the astigmatic mode converter is preserved on conversion to the LG_{nm} mode at the output? Using the definitions in Eqs. (1) and (2) it can be seen that M_r^2 as defined in Eqs. (13) and (14) satisfies the selection rule that M_r^2 is the same for both a HG_{nm} mode and a LG_{nm} mode with the same mode numbers (n,m). However, M_r^2 characterizes only the radial coordinate and not the entire transverse phase space area, it gives information about the average behavior of the transverse phase space. It is important to obtain detailed information on the all beam parameters which are characterized by a full set of second moments $(\overline{x^2}, \overline{y^2}, \overline{xy}, \overline{k_x^2}, \overline{k_y^2}, \overline{k_x k_y}, \overline{xk_x}, \overline{xk_y}, \overline{yk_x}$, and $\overline{yk_y})$ and to determine the total phase space area occupied by both the HG_{nm} and the LG_{nm} modes.

In the next section we consider a detailed characterization of the beam phase space area for each type of mode and discuss it's connection to the beam quality factor. It will be seen that M_r^2 will appear in the diagonal elements of the second moment matrix for a LG_{nm} but M_r^2 alone is not sufficient information to obtain the complete phase space area for a LG_{nm} mode which has off-diagonal elements in the second moment matrix.

4. Invariant phase space area

Following Ref. [17] we introduce the 4×4, real, symmetric matrix of the second order moments of the positions and wave vectors:

$$\mathrm{P}_{nm}^{\mathrm{HG,LG}} = \begin{pmatrix} \overline{x^2} & \overline{xy} & \overline{xk_x} & \overline{xk_y} \\ \overline{xy} & \overline{y^2} & \overline{yk_x} & \overline{yk_y} \\ \overline{xk_x} & \overline{yk_x} & \overline{k_x^2} & \overline{k_x k_y} \\ \overline{xk_y} & \overline{yk_y} & \overline{k_x k_y} & \overline{k_y^2} \end{pmatrix}, \tag{15}$$

where $\overline{x^2} \equiv \int\int x^2 |u_{nm}^{\mathrm{HG,LG}}(x,y,z=0)|^2 \, \mathrm{d}x \, \mathrm{d}y$ with similar expressions for $\overline{xy}$ and $\overline{y^2}$. Note that we need only evaluate these moments at the waist $z=0$ and we have assumed all of the first order moments vanish. Taking the Fourier transform of $u_{nm}^{\mathrm{HG,LG}}(x,y,z=0)$ allows us to calculate $\overline{k_x^2} \equiv \int\int k_x^2 |\tilde{u}_{nm}^{\mathrm{HG,\,LG}}(k_x,k_y,\ z=0)|^2 \mathrm{d}k_x \, \mathrm{d}k_y$, again there will be similar expressions for $\overline{k_x k_y}$ and $\overline{k_y^2}$.

The remaining off-diagonal moments require the density in both positions and wave vectors simultaneously. This is precisely the role played by the Wigner distribution function [6,8,18–20]

$$\mathrm{W}_{nm}^{\mathrm{HG,LG}}(x,y,k_x,k_y) \equiv \int\int u_{nm}^{\mathrm{HG,LG}}\left(x+\frac{a}{2}, y+\frac{b}{2}\right) u_{nm}^{\mathrm{HG,LG}\,*}\left(x-\frac{a}{2}, y-\frac{b}{2}\right) \cdot \exp\left[-\mathrm{i}\left(k_x a + k_y b\right)\right] \mathrm{d}a \, \mathrm{d}b. \tag{16}$$

We will not reproduce here the complicated expressions for the Wigner distribution functions for Hermite–Gauss and Laguerre–Gauss modes, which can be found in Refs. [19,20]. A typical moment is then computed as:

$$\overline{yk_x} \equiv \int\int\int\int y k_x W_{nm}^{\mathrm{HG,LG}}(x,y,k_x,k_y) \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}k_x \, \mathrm{d}k_y. \tag{17}$$

Note that all the moments can be computed directly from the Wigner function as its marginal distributions are $|u_{nm}^{\mathrm{HG,LG}}(x,y,z=0)|^2$ and $|u_{nm}^{\mathrm{HG,LG}}(k_x,k_y,z=0)|^2$.

The P matrix is the generalization that is required to compute the two dimensional phase space area when there is coupling between the various factors (x,y,k_x,k_y) resulting in the off-diagonal terms. We define the phase space area (PSA) as

$$\mathrm{PSA}_{nm}^{\mathrm{HG,LG}} \equiv \sqrt{\det\left[\mathrm{P}_{nm}^{\mathrm{HG,LG}}\right]}\,. \tag{18}$$

The P matrix as defined has the useful property that its determinant is invariant under rotations about the z axis [21]. We are then free to orient our (x,y) coordinate system in a fashion that simplifies the computation of the moments. Thus to compute the PSA for our input Hermite–Gauss mode when can ignore the fact that it is rotated by 45° which greatly simplifies the results. Note that this makes physical sense, as simply rotating the coordinate system should not change the phase space area.

For the Hermite–Gauss modes as defined in Eq. (1), all the off-diagonal terms in the P matrix vanish yielding

$$P_{nm}^{\mathrm{HG}} = \begin{pmatrix} (2n+1)\dfrac{w^2}{4} & 0 & 0 & 0 \\ 0 & (2m+1)\dfrac{w^2}{4} & 0 & 0 \\ 0 & 0 & (2n+1)\dfrac{1}{w^2} & 0 \\ 0 & 0 & 0 & (2m+1)\dfrac{1}{w^2} \end{pmatrix}. \tag{19}$$

The phase space area is then simply given as

$$\mathrm{PSA}_{nm}^{\mathrm{HG}} = \sqrt{\det\left[P_{nm}^{\mathrm{HG}}\right]} = \sqrt{\overline{x^2}\,\overline{k_x^2}\,\overline{y^2}\,\overline{k_y^2}} = \frac{(2n+1)(2m+1)}{4}. \tag{20}$$

Note that in this case where all the off-diagonal terms vanish, our generalized definition of the PSA reduces to the one given earlier in Eq. (9). Both the P matrix and the PSA can be written in terms of the beam quality factors defined in Eq. (12):

$$P_{nm}^{\mathrm{HG}} = \begin{pmatrix} M_x^2\dfrac{w^2}{4} & 0 & 0 & 0 \\ 0 & M_y^2\dfrac{w^2}{4} & 0 & 0 \\ 0 & 0 & M_x^2\dfrac{1}{w^2} & 0 \\ 0 & 0 & 0 & M_y^2\dfrac{1}{w^2} \end{pmatrix}, \tag{21a}$$

$$\mathrm{PSA}_{nm}^{\mathrm{HG}} = \sqrt{\det\left[P_{nm}^{\mathrm{HG}}\right]} = \frac{M_x^2 M_y^2}{4}. \tag{21b}$$

As required, we have recovered the physical significance of M_x^2 and M_y^2 as the measures of the increase in the phase space area for the higher order modes as compared to the fundamental Gaussian mode HG_{00}.

For the Laguerre–Gauss modes as defined in Eq. (2), the situation is more complicated as not all the off diagonal terms in the P matrix vanish. Using our earlier definitions, $p \equiv \min(n,m)$ and $\ell \equiv n-m$, the P matrix for the Laguerre–Gauss modes can be written as

$$\mathrm{P}_{nm}^{\mathrm{LG}} = \begin{pmatrix} (2p+|\ell|+1)\dfrac{w^2}{4} & 0 & 0 & \dfrac{\ell}{2} \\ 0 & (2p+|\ell|+1)\dfrac{w^2}{4} & -\dfrac{\ell}{2} & 0 \\ 0 & -\dfrac{\ell}{2} & (2p+|\ell|+1)\dfrac{1}{w^2} & 0 \\ \dfrac{\ell}{2} & 0 & 0 & (2p+|\ell|+1)\dfrac{1}{w^2} \end{pmatrix}. \quad (22)$$

The phase space area in this case is

$$\mathrm{PSA}_{nm}^{\mathrm{LG}} = \sqrt{\det\left[\mathrm{P}_{nm}^{\mathrm{LG}}\right]} = \frac{(2p+|\ell|-\ell+1)(2p+|\ell|+\ell+1)}{4} = \frac{(2n+1)(2m+1)}{4}. \quad (23)$$

We arrived at the desired result that the phase space area of the 45° rotated HG_{nm} mode and the LG_{nm} mode with the same mode numbers (n,m) are identical. As noted in the introduction, this is as it should be because the cylindrical lenses that make up the mode converter are symplectic elements. The selection rule that the mode numbers (n,m) for the input HG_{nm} mode and the output LG_{nm} mode be the same is dictated by the conservation of phase space area.

Note that Eqs. (22) and (23) can be written in terms of M_r^2 as follows:

$$\mathrm{P}_{nm}^{\mathrm{LG}} = \begin{pmatrix} M_r^2\dfrac{w^2}{4} & 0 & 0 & \dfrac{\ell}{2} \\ 0 & M_r^2\dfrac{w^2}{4} & -\dfrac{\ell}{2} & 0 \\ 0 & -\dfrac{\ell}{2} & M_r^2\dfrac{1}{w^2} & 0 \\ \dfrac{\ell}{2} & 0 & 0 & M_r^2\dfrac{1}{w^2} \end{pmatrix}, \quad (24a)$$

$$\mathrm{PSA}_{nm}^{\mathrm{LG}} = \sqrt{\det\left[\mathrm{P}_{nm}^{\mathrm{HG}}\right]} = \frac{\left(M_r^4-\ell^2\right)}{4}. \quad (24b)$$

It can be seen that M_r^2 is useful for characterizing the diagonal moments but provides no information on the off diagonal elements. As such M_r^2 only gives a complete characterization of the phase space area when $\ell=0$.

Table 1
Comparison of second moments for HG_{nm} and LG_{nm} modes to the fundamental Gaussian Mode HG_{00}

Diagonal moments	HG_{nm}	LG_{nm}
x_{nm}^2/x_{00}^2	$2n+1=M_x^2$	$2p+\lvert\ell\rvert+1=M_r^2$
y_{nm}^2/y_{00}^2	$2m+1=M_y^2$	$2p+\lvert\ell\rvert+1=M_r^2$
$k_{x,nm}^2/k_{x,00}^2=\theta_{x,nm}^2/\theta_{x,00}^2$	$2n+1=M_x^2$	$2p+\lvert\ell\rvert+1=M_r^2$
$k_{y,nm}^2/k_{y,00}^2=\theta_{y,nm}^2/\theta_{y,00}^2$	$2m+1=M_y^2$	$2p+\lvert\ell\rvert+1=M_r^2$

For an arbitrary light beam, the full 4×4 matrix of second order moments is required and the phase space area occupied by the light beam is given by Eq. (18). If a generalized beam quality factor is desired it should be defined as follows:

$$M_{\mathrm{tot}}^4 \equiv \frac{\text{Phase space area for an arbitrary beam}}{\text{Minimum phase space area}} = \frac{\sqrt{\det[\mathrm{P}_{nm}]}}{1/4} = 4\sqrt{\det[\mathrm{P}_{nm}]}\,. \tag{25}$$

For a HG_{nm} mode, $M_{\mathrm{tot}}^4 = M_x^2 \cdot M_y^2$ and for a LG_{nm} mode $M_{\mathrm{tot}}^4 = M_r^4 - \ell^2$.

5. Discussion of the diagonal moments

In Table 1 we compare the diagonal terms of the P matrix for general HG_{nm} and LG_{nm} modes to the corresponding value for the fundamental mode $\mathrm{HG}_{00} = \mathrm{LG}_{00}$. This is done to emphasis the fact that although all the modes have the same waist parameter w_0, this parameter does not characterize the spread in transverse positions of the higher order modes at the waist or the spread in far field divergence angles. As pointed out in [22,23], the spread in both positions (x,y) and far-field divergence angles (θ_x,θ_y) is best characterized by the second moments of these quantities and they increase as the mode numbers (n,m) increase. The relevant expressions are listed in Table 1. It can be seen that the spread in both the position and far-field angle is an increasing function of the mode numbers (n,m).

6. Concluding remarks

In this paper we have considered the second order moments and the phase space area of Hermite–Gauss and Laguerre–Gauss modes. We have shown that the principle of conservation of phase space area is responsible for the selection rule governing the conversion of Hermite–Gauss modes into Laguerre–Gauss modes using the astigmatic mode converter described in Refs. [1–4]. Finally we have discussed how these results are related to beam quality factors in common use to characterize light beams and the generalization of these beam quality factors to arbitrary light beams.

Acknowledgements

The author is grateful to Prof. A.E. Siegman of Stanford University for providing reprints of his articles and for maintaining the very useful WWW based bibliography of papers related to beam quality.

References

[1] E. Abramochkin, V. Volstnikov, Opt. Commun. 83 (1991) 123.
[2] M.W. Beijersbergen, L. Allen, H.E.L.O. van der Veen, J.P. Woerdman, Opt. Commun. 96 (1993) 123.
[3] M.J. Padgett, J. Arlt, N. Simpson, Am. J. Phys. 64 (1996) 77.
[4] J. Courtial, M.J. Padgett, Opt. Commun. 159 (1999) 13.
[5] A.S. Davydov, Quantum Mechanics, Pergamon, 1965, p. 37.
[6] Y. S. Kim, M.E. Noz, Phase Space Picture of Quantum Mechanics: Group Theoretical Approach, World Scientific, 1991.
[7] D. Gloge, D. Marcuse, J. Opt. Soc. Am. 59 (1969) 1629.
[8] M.J. Bastiaans, Optik 82 (1989) 173.
[9] A.E. Siegman, SPIE J. 1224 (1990) 2.
[10] P.A. Belanger, Opt. Lett. 16 (1991) 196.

[11] M.W. Sasnett, T.F. Johnston, SPIE J. 1414 (1991) 21.
[12] D.R. Whitehouse, SPIE J. 1834 (1992) 86.
[13] G.N. Lawrence, Laser Focus World, 1994, p. 109.
[14] K. Witting, A. Giesen, H. Hugel, Appl. Opt. 33 (1994) 3837.
[15] Th. Graf, J.E. Balmer, Opt. Commun. 131 (1996) 77.
[16] S. Saghafi, C.J.R. Sheppard, Opt. Commun. 153 (1998) 207.
[17] G. Nemes, A.E. Siegman, J. Opt. Soc. Am. 11 (1994) 2257.
[18] E. Wigner, Phys. Rev. 40 (1932) 749.
[19] R. Gase, Opt. Quantum Electron. 24 (1992) S1109.
[20] R. Gase, IEEE J. Quantum Electron. 31 (1995) 1811.
[21] R. Martinez-Herrero, P.M. Meijas, M. Arias, Opt. Lett. 20 (1995) 651.
[22] W.H. Carter, Appl. Opt. 19 (1980) 1027.
[23] R.L. Phillips, L.C. Andrews, Appl. Opt. 22 (1983) 643.